

What is a horocyclic product, and how is it related to lamplighters?

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This is a rather personal introductory outline of an interesting class of geometric, resp. graph- & group-theoretical structures. After an introductory section about their genesis, the general construction of horocyclic products is presented. Three closely related basic structures of this type are explained in more detail: Diestel-Leader graphs, treebolic spaces, and Sol-groups, resp. -manifolds. Emphasis is on their geometry, isometry groups, quasi-isometry classification and boundary at infinity. Subsequently, it is clarified under which parametrisation they admit discrete groups of isometries acting with compact quotient. Finally, further developments are reviewed briefly.

1 A problem on infinite graphs

In the mid-1980ies, in conversations with my colleagues at Leoben, I repeatedly asked the following question:

Are there any vertex-transitive graphs that do not look like Cayley graphs?

The drawback was that I didn't see how to define "look like" rigorously. My eyes were opened when I encountered GROMOV's definition of *quasi-isometry* in [23], resp. (more clearly) [24, 7.2.G].

Before proceeding, we should clarify the involved notions and start a preliminary discussion. A *graph* will be written in terms of its vertex set X , which carries

a symmetric neighbourhood relation \sim . Thus, the edges are pairs $[x, y] = [y, x]$, where $x \sim y$, so that we allow loops $[x, x]$, but no multiple edges. Usually, our graphs will be *infinite*. The *degree* $\deg(x)$ of $x \in X$ is the number of neighbours. Everybody is familiar with the concept of a *path* $[x_0, x_1, \dots, x_n]$ in a graph: one has to have $x_k \sim x_{k-1}$, and the length of a path is its number of edges (here: n).

All our graphs will be *connected* (for all $x, y \in X$ there is a path starting at x and ending at y) and *locally finite* ($\deg(x) < \infty$ for every x). Being connected, X becomes a metric space, where the graph distance $d(x, y)$ is the minimal length of a path from x to y .

An *automorphism* X is a self-isometry of (X, d) . We write $\text{Aut}(X)$ for the group of all automorphisms of X . The graph is called *vertex-transitive* if for every $x, y \in X$ there is $g \in \text{Aut}(X)$ such that $gx = y$. A large class of such graphs is provided by groups: given a finitely generated group G (usually written multiplicatively) and a finite, symmetric set S of generators, we can visualise G by its *Cayley graph* $X(G, S)$. Its vertex set is $X = G$, and $x \sim y$ if $y = xs$ for some $s \in S$ (so that $y = xs^{-1}$). The group acts by automorphisms on $X(G, S)$ via $(g, x) \mapsto gx$. The most typical examples are

1. The Cayley graph of the additive group \mathbb{Z}^2 with respect to $S = \{(\pm 1, 0), (0, \pm 1)\}$ – this is the square lattice;
2. The Cayley graph of the free group \mathbb{F}_2 on two free generators a, b with respect to $S = \{a^{\pm 1}, b^{\pm 1}\}$ – this is the homogeneous tree with degree 4.

See Figure 1. Furthermore, the homogeneous tree with arbitrary degree $p + 1$ is also the Cayley graph of the group $\langle a_1, \dots, a_{p+1} : a_i^2 = 1_G \rangle$.

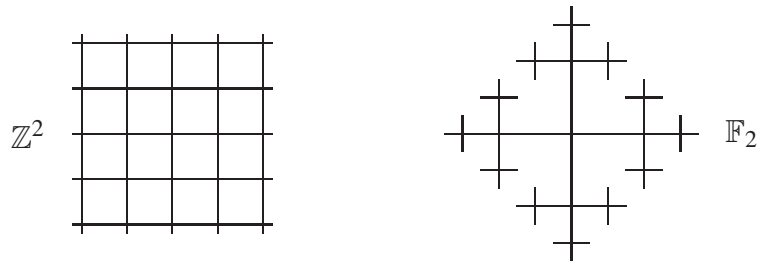


Figure 1

There are vertex-transitive graphs which are not Cayley graphs. A finite example is the well-known Petersen graph. From here one can of course construct infinite examples (e.g. the Cartesian product of the Petersen graph with the bi-infinite line). But there also are intrinsically infinite examples of non-Cayley vertex-transitive graphs. One of them is based on the following way of looking at trees, which will play an important role later on: take the homogeneous tree \mathbb{T}_p with

degree $p + 1$, but draw it differently, such that it “hangs down” from a point ∞ at infinity. See Figure 2, where $p = 2$. That is, the tree is considered as the union of generations (horizontal layers) – called *horocycles* – H_k , $k \in \mathbb{Z}$. Each H_k is infinite, every vertex $x \in H_k$ has a unique neighbour in H_{k-1} , its *predecessor* x^- , and p neighbours in H_{k+1} , its *successors*. Thus, p is the *branching number* of \mathbb{T} . For $x \in \mathbb{T}$, we write $\mathfrak{h}(x) = k$ if $x \in H_k$, the *Busemann function*. An ancestor of x is an iterated predecessor. Any pair of vertices x, y has a common ancestor v for which $\mathfrak{h}(v)$ is maximal. We write $v = x \wedge y$. Also, we choose a root vertex o in H_0 .

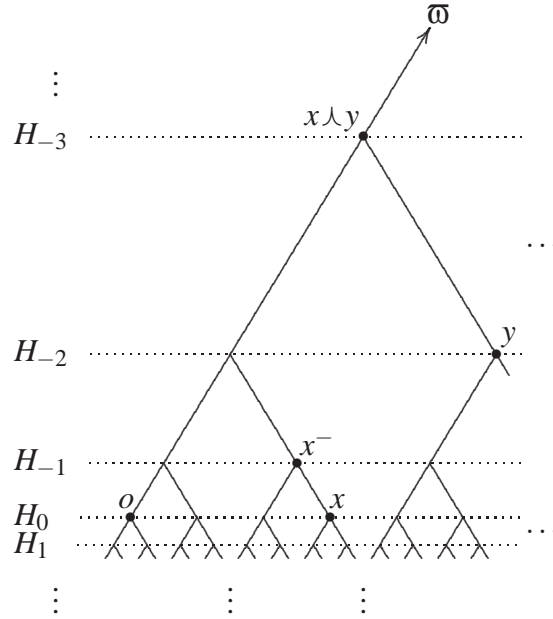


Figure 2

Consider the group $\text{Aff}(\mathbb{T})$ of all automorphisms g of \mathbb{T} which preserve the predecessor relation: $g(x^-) = (gx)^-$ for all x . It acts transitively. It is called the affine group of the tree because it contains the group of all affine mappings $\xi \mapsto \alpha\xi + \beta$ of the ring \mathbb{Q}_p of p -adic numbers (field, if p is prime), where $\xi, \alpha, \beta \in \mathbb{Q}_p$ and α is invertible, see CARTWRIGHT, KAIMANOVICH AND WOESS [11, §4]. In particular, \mathbb{Q}_p can be identified with the lower boundary $\partial^*\mathbb{T}_p$ that we are going to describe further below.

Next, we introduce the additional edges $[x, (x^-)^-]$ for all x , see Figure 3. The resulting graph is sometimes called the *grandmother graph*, which is suggestive when one thinks of \mathbb{T} as an infinite genealogical tree.

The point is that $\text{Aff}(\mathbb{T})$ now becomes the full automorphism group of the grandmother graph.

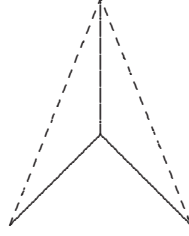


Figure 3

(1.1) Claim. *The grandmother graph is vertex-transitive, but not a Cayley graph of some finitely generated group.*

How does one prove that a given graph is or is not a Cayley graph?

(1.2) Criterion. *Let X be a locally finite, connected graph and G be a subgroup of $\text{Aut}(X)$. Then X is a Cayley graph of G if and only if G acts on X transitively and with trivial vertex-stabilisers.*

Here the stabiliser of $x \in G$ is of course $G_x = \{g \in G : gx = x\}$, and “trivial” means that it consists only of the identity.

Now assume that a group of automorphisms G acts transitively on the grandmother graph. Then $G \leq \text{Aff}(\mathbb{T})$. Let x be a vertex and y, z be two of its successors. Then there must be $g \in G$ such that $gy = z$, so that $g \neq \text{id}$. But we must have $gy^- = z^-$, that is $gx = x$. Thus, G_x is non-trivial, which proves Claim 1.1.

However, everybody will agree that the grandmother graph looks (vaguely) like the tree itself, which *is* a Cayley graph. So from the point of view of the initial question, this is not yet a satisfactory example. Let us now come to the definition of “look like”.

(1.3) Definition. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $\phi : X_1 \rightarrow X_2$ is called a quasi-isometry, if there are constants $A > 0$ and $B \geq 0$ such that for all $x_1, y_1 \in X_1$ and $x_2 \in X_2$,*

$$(i) \quad d_2(x_2, \phi x_1) \leq B \quad (\text{quasi-surjective}), \text{ and}$$

$$(ii) \quad \frac{1}{A} d_2(\phi x_1, \phi x_2) - B \leq d_1(x_1, y_1) \leq A d_2(\phi x_1, \phi x_2) + B \quad (\text{quasi-bi-Lipschitz}).$$

If $B = 0$, the mapping is called bi-Lipschitz.

Every quasi-isometry ϕ has a quasi-inverse $\phi^* : X_2 \rightarrow X_1$, i.e., a quasi-isometry such that $\phi^* \phi$ and $\phi \phi^*$ are bounded perturbations of the identity on X_1 , resp.

X_2 (i.e., the image of any element is at bounded distance). In particular, quasi-isometry is an equivalence relation.

Any two Cayley graphs of a group with respect to different, finite symmetric sets of generators are bi-Lipschitz. After being promoted by Gromov, the study of quasi-isometry invariants of finitely generated groups has become a “big business” which is at the core of what is since then called Geometric Group Theory (in good part replacing the earlier name “Combinatorial Group Theory”).

The identity map on the vertex set is a bi-Lipschitz mapping between the grand-mother graph and the tree, so that we have a non-Cayley vertex transitive graph which is quasi-isometric with a Cayley graph. My question now could be formulated rigorously as follows.

Is there a (connected, locally finite, infinite) vertex-transitive graph that is not quasi-isometric with some Cayley graph?

I posed this question explicitly in [30] and [33] (published in 1990, resp. 1991). This appeared to be a difficult problem, and I learnt that there has to be a positive correlation between the difficulty of a mathematical question and the fame of the person who poses it. Initially, geometric group theorists ignored my problem or even made fun of it. However, in the world of Graph Theory, there is an exclusive minority interested in infinite graphs, and in the mid-early 1990ies, DIESTEL AND LEADER came up with a construction of a graph which they believed to provide the answer to the question. This was what I later called the Diestel-Leader graph $DL(2, 3)$, whose construction will be explained in a moment. However, it resisted their and my efforts (as well as the efforts of several visitors of mine who were involved in this discussion) to prove that it was indeed not quasi-isometric with any Cayley graph. At last, in 2001, Diestel and Leader made their construction and conjecture public without a proof [16].

Let us now describe the construction. We take two trees \mathbb{T}_p and \mathbb{T}_q with respective branching numbers p and q (not necessarily distinct). We look at each of them as in Figure 2, but the second tree is upside down. On each of them, we have the respective Busemann function \mathfrak{h} . (We omit putting an index.)

(1.4) Definition. *The Diestel-Leader graph $DL(p, q)$ is*

$$DL(p, q) = \{(x_1, x_2) \in \mathbb{T}_p \times \mathbb{T}_q : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\},$$

and neighbourhood is given by

$$(x_1, x_2) \sim (y_1, y_2) \iff x_1 \sim y_1 \text{ and } x_2 \sim y_2.$$

Thus, either $x_1^- = y_1$ and $y_2^- = x_2$ or vice versa.

To visualize $\text{DL}(p, q)$, draw \mathbb{T}_p in horocyclic layers as in Figure 2, and right to it \mathbb{T}_q in the same way, but upside down, with the respective horocycles $H_k(\mathbb{T}_p)$ and $H_{-k}(\mathbb{T}_q)$ on the same level. Connect the two origins o_1, o_2 by an elastic spring. It is allowed to move along each of the two trees, may expand infinitely, but must always remain in horizontal position. The vertex set of $\text{DL}(p, q)$ consists of all admissible positions of the spring. From a position (x_1, x_2) with $\mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0$ the spring may move downwards to one of the q successors of x_2 in \mathbb{T}_q , and at the same time to the predecessor of x_1 in \mathbb{T}_p , or it may move upwards in the analogous way. Such a move corresponds to going to a neighbour of (x_1, x_2) . Figure 2 depicts $\text{DL}(2, 2)$.

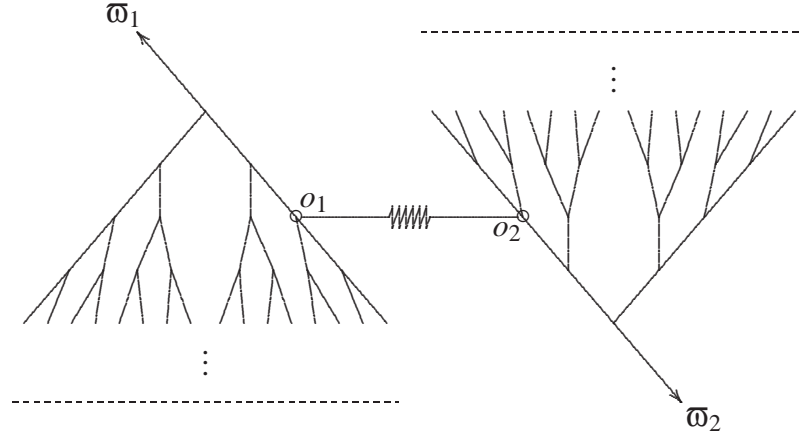


Figure 4

We first explain that $\text{DL}(p, q)$ is vertex-transitive, but when $p \neq q$, it is not a Cayley graph. Recall the group $\text{Aff}(\mathbb{T})$ of all automorphisms that preserve the predecessor relation, where $\mathbb{T} = \mathbb{T}_p$ or \mathbb{T}_q . One easily verifies (see CARTWRIGHT, KAIMANOVICH AND WOESS [11]) that the mapping

$$\Phi : \text{Aff}(\mathbb{T}) \rightarrow \mathbb{Z}, \quad \Phi(g) = \mathfrak{h}(gx) - \mathfrak{h}(x) \quad (1.5)$$

is independent of $x \in \mathbb{T}$, and thus a homomorphism onto the additive group \mathbb{Z} . That is, every $g \in \text{Aff}(\mathbb{T})$ shifts the tree up or down by the vertical amount $\Phi(g)$. Now the following is not hard to prove.

(1.6) Proposition. *The group*

$$\mathcal{A} = \mathcal{A}(p, q) = \{(g_1, g_2) \in \text{Aff}(\mathbb{T}_p) \times \text{Aff}(\mathbb{T}_q) : \Phi(g_1) + \Phi(g_2) = 0\}$$

acts transitively on $\text{DL}(p, q)$ by

$$(x_1, x_2) \mapsto (g_1 x_1, g_2 x_2).$$

If $p \neq q$, then this is the full automorphism group of $\text{DL}(p, q)$, while when $p = q$, then it has index 2 in the full automorphism group, which is generated by \mathcal{A} and the “reflection” $(x_1, x_2) \mapsto (x_2, x_1)$.

(The vertex set of) $\text{DL}(p, q)$ is the disjoint union of the horoplans

$$H_{k,-k} = \{(x_1, x_2) \in \mathbb{T}_p \times \mathbb{T}_q : \mathfrak{h}(x_1) = k, \mathfrak{h}(x_2) = -k\}$$

and every $g = (g_1, g_2) \in \mathcal{A}$ maps $H_{k,-k}$ to $H_{m,-m}$, where $m = k + \Phi(g_1)$.

(1.7) Lemma. *If $q \neq p$ then $\text{DL}(p, q)$ is not a Cayley graph of some finitely generated group.*

Proof. Suppose that $q > p$, and that G is any group of automorphisms that acts transitively on $\text{DL}(p, q)$. Consider the sets $A = \{(o_1, x_2) : x_2^- = o_2^-\} \subset H_{0,0}$ and $B = \{(x_1, o_2^-) : x_1^- = o_1\} \subset H_{1,-1}$. Then $|A| = q$, $|B| = p$, and the subgraph of $\text{DL}(p, q)$ induced by $A \cup B$ is the complete bipartite graph over A and B (there is an edge between each element of A and each element of B). The set B consists of all neighbours of A in $H_{1,-1}$.

For each $x = (o_1, x_2) \in A$ there must be $g_x \in G$ such that $g_x o = x$, where $o = (o_1, o_2)$. Since $G \subset \mathcal{A}$, each g_x sends every horoplane to itself, and preserves neighbourhood. We conclude that each g_x sends B onto itself. But since $|B| < |A|$, there must be two distinct $x, x' \in A$ and $y \in B$ such that $g_x y = g_{x'} y$. Then $g_x^{-1} g_{x'}$ stabilises y , although it is different from the identity. In view of Criterion 1.2, $\text{DL}(p, q)$ cannot be a Cayley graph of G . \square

The last proof gives a clue why $\text{DL}(p, q)$ should not be quasi-isometric with some Cayley graph, when $p \neq q$: briefly spoken, our graph grows on the order of p^n in one vertical direction, and of order q^n in the opposite direction.

The result was finally announced in 2007 by a group of quasi-isometry experts, ESKIN, FISHER AND WHYTE [17], and the proof is contained in the first of the two papers [18], [19] within a more general framework of quasi-isometry classification of structures whose construction is very similar to DL-graphs. Thus, at last, my question made it to the *Annals*:

(1.8) Theorem. [18]. *If $q \neq p$ then $\text{DL}(p, q)$ is not quasi-isometric with any finitely generated group.*

2 Horocyclic products

We now explain the first of the two notions of the title of this article. Let X be a metric space. A *level function* or *Busemann function* is a continuous surjection $\mathfrak{h} : X \rightarrow \mathbb{L}$, where $\mathbb{L} = \mathbb{R}$, or when X is discrete, resp. totally disconnected, $\mathbb{L} = \mathbb{Z}$. We write H_l ($l \in \mathbb{L}$) for the associated level sets, i.e., the preimages under \mathfrak{h} of l . We call them *horocycles* or *horospheres*.

Usually, our X will carry additional structure, and then the function \mathfrak{h} should be adapted to that structure. If X is a (connected) graph, then it has to be a graph homomorphism (neighbourhood preserving surjection) onto \mathbb{Z} , the latter seen as the bi-infinite line graph. In particular, edges of X are only allowed between successive horocycles.

If $X = G$ is a discrete (or more generally, totally disconnected) group, then we will need \mathfrak{h} to be a group homomorphism onto \mathbb{Z} , while if it is a connected locally compact group, it has to be a (continuous) homomorphism onto \mathbb{R} . (More general choices of Abelian groups \mathbb{L} also work, but will not be considered here.)

We refer to (X, \mathfrak{h}) as a *Busemann pair* over \mathbb{L} , although this expression is justified only in specific cases.

(2.1) Definition. Let (X_1, \mathfrak{h}_1) and (X_2, \mathfrak{h}_2) be two Busemann pairs over the same \mathbb{L} . We shall commonly use the same symbol \mathfrak{h} for both \mathfrak{h}_i . The horocyclic product of X_1 and X_2 is

$$X_1 \times_{\mathfrak{h}} X_2 = \{(x_1, x_2) \in X_1 \times X_2 : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$$

(On some occasions it may be more natural to require that $\mathfrak{h}(x_1) - \mathfrak{h}(x_2) = 0$.)

In general, $X_1 \times_{\mathfrak{h}} X_2$ is a topological subspace of the direct product space $X_1 \times X_2$. In the group case, it is a normal subgroup of the direct product. In the graph case, as edges in the X_i may occur only between successive horocycles, $X_1 \times_{\mathfrak{h}} X_2$ is an induced subgraph of the direct product of the two graphs. That is,

$$(x_1, x_2) \sim (y_1, y_2) \iff x_i \sim y_i \ (i = 1, 2), \quad \text{and then} \\ \mathfrak{h}(x_1) - \mathfrak{h}(y_1) = \mathfrak{h}(y_2) - \mathfrak{h}(x_2) = \pm 1.$$

(2.2) Remark. It may also be good to consider graphs as one-dimensional complexes, where each edge is a copy of the unit interval. The graph metric extends naturally to the interior points of the edges. If we have a \mathbb{Z} -valued Busemann function \mathfrak{h} on the vertex set, and $e = [x, y]$ is an edge with $x \in H_k$ and $y \in H_{k+1}$, then we can extend \mathfrak{h} to every interior point $z \in e$: if $d(z, x) = \kappa \in [0, 1)$, then $\mathfrak{h}(z) = k + \kappa$.

In this way, the Busemann function becomes \mathbb{R} -valued, and the topology of the resulting horocyclic product yields just the one-dimensional complex that one gets from the graph construction with edges \equiv intervals.

Three sister structures

We now consider three families of spaces. The fact that they share many common geometric features becomes apparent by realising that they all are horocyclic products:

- A. horocyclic product of two trees \rightarrow Diestel-Leader graphs;
- B. horocyclic product of a hyperbolic half-plane and a tree \rightarrow treebolic spaces;
- C. horocyclic product of two hyperbolic half-planes \rightarrow Sol-groups, resp. manifolds.

Thinking of a graph as a 1-complex as in Remark 2.2, our structures are 1-dimensional in A, 2-dimensional in B, and 3-dimensional in C.

A. More on Diestel-Leader graphs

We start with some general observations. The automorphism group of any locally finite, connected graph X carries the topology of pointwise convergence (on the vertex set), and as such, it is a locally compact, totally disconnected group. See e.g. TROFIMOV [31], or [33]. Let G be any closed subgroup of $\text{Aut}(X)$ that acts transitively. It has a left Haar measure λ_G (unique up to multiplication with a constant), and there is the *modular function* Δ_G defined by $\Delta_G(g) = \lambda_G(Ug)/\lambda_G(U)$, where $U \subset G$ is open with compact closure. $\Delta_G(g)$ is independent of the choice of U , and we may take $U = G_x$, the stabiliser of some vertex x . The group is called unimodular when $\Delta_G \equiv 1$.

(2.3) Lemma. [29], [31]. *If $g \in G$ and $gx = y$ then $\Delta_G(g) = |G_o x|/|G_x o|$.*

A connected graph X with bounded vertex degrees is called *amenable*, if

$$\inf\{|\partial F|/|F| : F \subset X \text{ finite}\} = 0.$$

A non-amenable graph is sometimes called *infinite expander*. A locally compact group is called *amenable*, if it carries a left-invariant mean m , that is, a finitely additive measure that satisfies $m(G) = 1$ and $m(gU) = m(U)$ for any $g \in G$ and Borel set $U \subset G$. The following is due to SOARDI AND WOESS [30].

(2.4) Proposition. *A vertex-transitive graph X is amenable if and only if some (\iff every) closed subgroup G of $\text{Aut}(X)$ that acts transitively is both amenable and unimodular.*

We also note that a horocyclic product of two amenable groups is amenable, since it is a subgroup of the direct product of the two groups. (Known fact: closed subgroups as well as direct products of amenable groups are amenable.) Now it is easy to see and well-known that $\text{Aff}(\mathbb{T}_p)$ is an amenable group, see e.g. [34, Lemma 12.14], and its modular function is $\Delta_{\text{Aff}(\mathbb{T}_p)}(g) = p^{\Phi(g)}$. The group $\mathcal{A}(p, q)$ of Proposition 1.6 is the horocyclic product of $\text{Aff}(\mathbb{T}_p)$ and $\text{Aff}(\mathbb{T}_q)$ with respect to the respective Busemann functions $\mathfrak{h}(g) = \Phi(g)$, where Φ is given by (1.5).

It is easy to compute the modular function of the group $\mathcal{A}(p, q)$.

(2.5) Corollary. *The modular function of the group $\mathcal{A}(p, q)$ is given by*

$$\Delta_{\mathcal{A}}(g) = (q/r)^{\Phi(g)},$$

where $\Phi(g) = \Phi(g_1) = -\Phi(g_2)$ for $g = g_1 g_2 \in \mathcal{A}(p, q)$.

Thus, the group $\mathcal{A}(p, q)$ is unimodular, and the graph $\text{DL}(p, q)$ is amenable if and only if $p = q$.

This leads to another view on the fact that $\text{DL}(p, q)$ is not a Cayley graph when $p \neq q$. Indeed, more generally, when $p \neq q$, then there cannot be a finitely generated group of automorphisms that acts on $\text{DL}(p, q)$ with finitely many orbits and finite vertex stabilisers: such a group would have to be a co-compact lattice, i.e., a discrete subgroup of $\mathcal{A}(p, q)$ with compact quotient, which cannot occur in a non-unimodular group. Further below, we shall see that when $p = q$, the Diestel-Leader graph is a Cayley graph.

Regarding the quasi-isometry classification, we quote another result of ESKIN, FISHER AND WHYTE.

(2.6) Theorem. [17]+[18]. *$\text{DL}(p, q)$ is quasi-isometric with $\text{DL}(p', q')$ if and only if p and p' are powers of a common integer, q and q' are powers of a common integer, and $\log p' / \log p = \log q' / \log q$.*

Another object whose description may be of interest is the geometric boundary at infinity of $\text{DL}(p, q)$.

For that purpose, we first need to describe the geometric boundary of an arbitrary infinite, locally finite tree T (not necessarily homogenous). For any x, y in T , there is a unique geodesic path $\pi(x, y) = [x_0, \dots, x_n]$ such that $d(x_i, x_j) = |i - j|$ for all i, j . Analogously, a geodesic ray, resp. (two-sided) geodesic is an infinite

path $\pi = [x_0, x_1, x_2, \dots]$, resp. $\pi = [\dots, x_{-1}, x_0, x_1, x_2, \dots]$, such that $d(x_i, x_j) = |i - j|$ for all i, j . We think of a ray as a way of going to a point at infinity. Then two rays describe the same point at infinity, i.e., they are equivalent, if their symmetric difference is finite. This means that they differ only by finite initial pieces. An *end* of T is an equivalence class of rays. The boundary ∂T is the set of all ends. For any $x \in T$ and $\xi \in \partial T$, there is a unique geodesic ray $\pi(x, \xi)$ that starts at x and represents ξ . For any pair of distinct ends ξ, η , there is a unique geodesic $\pi(\xi, \eta) = [\dots, x_{-1}, x_0, x_1, x_2, \dots]$ such that $[x_0, x_{-1}, x_{-2}, \dots]$ represents ξ and $[x_0, x_1, x_2, \dots]$ represents η .

We choose a reference point $o \in T$ and let $|x| = d(o, x)$ for $x \in T$. For $w, z \in \widehat{T} = T \cup \partial T$, we define their *confluent* $w \wedge z$ with respect to o by

$$\pi(o, w \wedge z) = \pi(o, w) \cap \pi(o, z).$$

This is a vertex, namely the last common element on the geodesics $\pi(o, w)$ and $\pi(o, z)$, unless $w = z \in \partial T$. We equip \widehat{T} with the following ultra-metric.

$$\theta(w, z) = \begin{cases} e^{-|w \wedge z|}, & \text{if } z \neq w, \\ 0, & \text{if } z = w. \end{cases}$$

Then \widehat{T} is compact, and T is open and dense. In the induced topology, a sequence $z_n \in \widehat{T}$ converges to $\xi \in \partial T$ if and only if $|z_n \wedge \xi| \rightarrow \infty$.

Back to \mathbb{T}_p , we choose a reference end $\varpi \in \partial \mathbb{T}_p$ and let $\partial^* \mathbb{T}_p$ be the remaining punctured boundary. In Figure 2, ϖ is at the top and $\partial^* \mathbb{T}_p$ at the bottom. The function \mathfrak{h} is indeed the Busemann function with respect to ϖ in the classical sense: for any vertex x ,

$$\mathfrak{h}(x) = \lim_{y \rightarrow \xi} (d(x, y) - d(o, y)) = d(x, x \wedge o) - d(o, x \wedge o),$$

where (recall) $x \wedge o$ is the maximal common ancestor of x and o , see Figure 2. (It is the confluent of x and o with respect to the end ϖ instead of the vertex o .)

In taking our two trees, we have two reference ends, $\varpi_1 \in \partial \mathbb{T}_p$ and $\varpi_2 \in \partial \mathbb{T}_q$, see Figure 4. Now we can describe the natural geometric compactification of $\text{DL}(p, q)$: it is a subgraph of $\mathbb{T}_p \times \mathbb{T}_q$, and the obvious geometric compactification of the latter product space is $\widehat{\mathbb{T}}_p \times \widehat{\mathbb{T}}_q$.

(2.7) Definition. *The geometric compactification $\widehat{\text{DL}}(p, q)$ is the closure of $\text{DL}(p, q)$ in $\widehat{\mathbb{T}}_p \times \widehat{\mathbb{T}}_q$, and the boundary at infinity is*

$$\partial \text{DL}(p, q) = \widehat{\text{DL}}(p, q) \setminus \text{DL}(p, q).$$

We can imagine the boundary as a “filled ultra-metric 8”. It is

$$\partial \text{DL}(p, q) = \left(\widehat{\mathbb{T}}_p \times \{\varpi_2\} \right) \cup \left(\{\varpi_1\} \times \widehat{\mathbb{T}}_q \right).$$

The two pieces meet in the point (ϖ_1, ϖ_2) , see Figure 5.

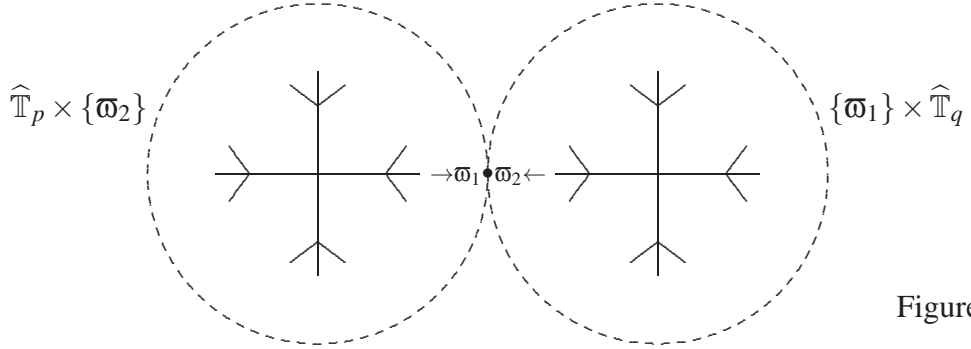


Figure 5

Here, the topology of the boundary dictates disk-like pictures of the two trees with their boundaries, while Figure 2 is an upper-half-plane-like picture.

Let us clarify convergence to the boundary of a sequence $\mathbf{x}_n = (x_{1,n}, x_{2,n}) \in \text{DL}(p, q)$ in the resulting topology. At least one of $x_{1,n}$ and $x_{2,n}$ has to converge to a boundary point of the respective tree. If $x_{1,n} \rightarrow \xi_1 \in \partial^* \mathbb{T}_p$, then necessarily $x_{2,n} \rightarrow \varpi_2$, whence $\mathbf{x}_n \rightarrow (\xi_1, \varpi_2)$. Analogously, if $x_{1,n} = x_1 \in \mathbb{T}_p$ for all $n \geq n_0$, then necessarily $x_{2,n} \rightarrow \varpi_2$, whence $\mathbf{x}_n \rightarrow (x_1, \varpi_2)$. In the same way, when $x_{2,n} \rightarrow \xi_2 \in \partial^* \mathbb{T}_p$, resp. $x_{2,n} = x_2 \in \mathbb{T}_q$ for all $n \geq n_0$, then $\mathbf{x}_n \rightarrow (\varpi_1, \xi_2)$, resp. $\mathbf{x}_n \rightarrow (\varpi_1, x_2)$. Finally, it is possible that $x_{1,n} \rightarrow \varpi_1$ and $x_{2,n} \rightarrow \varpi_2$ (for example by staying on a fixed horizontal level). In this case, $\mathbf{x}_n \rightarrow (\varpi_1, \varpi_2)$.

To conclude this description of the geometry of $\text{DL}(p, q)$, we display the formula for the graph metric, due to BERTACCHI [6].

(2.8) Lemma. *In $\text{DL}(p, q)$,*

$$d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2) - |\mathfrak{h}(x_1) - \mathfrak{h}(x_2)|.$$

B. Treebolic spaces

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ be the upper half plane with the hyperbolic metric

$$d(z_1, z_2) = \log \frac{|z_1 - \bar{z}_2| + |z_1 - z_2|}{|z_1 - \bar{z}_2| - |z_1 - z_2|}.$$

Recall that geodesics (shortest paths) lie on semi-circles orthogonal to the real axis, resp. vertical lines. The standard Busemann function with respect to the upper boundary point ∞ is $z \mapsto \log(\operatorname{Im} z)$. In comparing with the tree, the sign is reversed – it increases when going to ∞ . (This is related with the fact that the real and p -adic absolute values of p^n , $n \in \mathbb{Z}$, have opposite behaviour.) Now we rescale the Busemann function by choosing a *real* parameter $q > 1$ and setting $\mathfrak{h}(z) = \mathfrak{h}_q(z) = \log_q(\operatorname{Im} z)$. Among the resulting horocycles, there are the ones where $\mathfrak{h}_q(z) = k \in \mathbb{Z}$, that is, $\operatorname{Im} z = q^k$. Drawing these in the upper half plane yields to a picture to which we sometimes refer as *sliced hyperbolic plane* \mathbb{H}_q , see Figure 6.

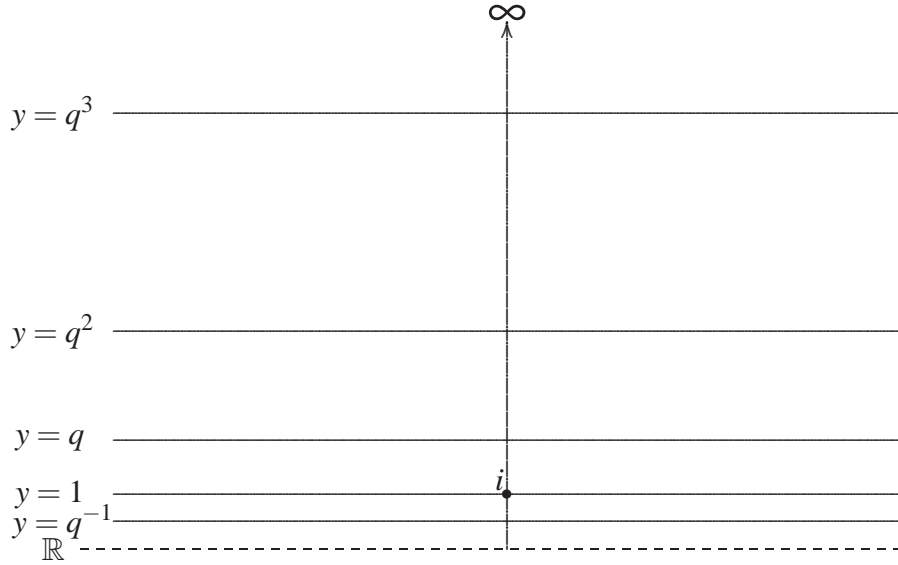


Figure 6

Now we look at the tree \mathbb{T}_p as in Figure 2, but upside down, so that ϖ is at the bottom, and the tree branches upwards. As in Remark 2.2, we consider it as a metric tree where edges are intervals of length 1, so that the Busemann function of the tree becomes real-valued. Then we can consider the horocyclic product with sliced hyperbolic plane. This is a situation where we pair points $z \in \mathbb{H}_q$ and $w \in \mathbb{T}_p$ when $\mathfrak{h}_q(z) - \mathfrak{h}(w) = 0$ with “ $-$ ” instead of “ $+$ ” because of the opposite behaviour of the two functions mentioned above.

(2.9) Definition. For integer $p \geq 2$ and real $q > 0$, treebolic space is defined as

$$\mathrm{HT}(p, q) = \{z = (w, z) \in \mathbb{T}_p \times \mathbb{H}_q : \mathfrak{h}(w) = \log_q(\operatorname{Im} z)\}.$$

In these terms, treebolic space was introduced – with notation $\mathrm{HT}(q, p)$ and elements (z, w) in the place of (w, z) – by BENDIKOV, SALOFF-COSTE, SALVATORI

AND WOESS [3] and studied in detail in [4]. Previously, $\text{HT}(p, p)$ (with integer $p \geq 2$) appeared in the work of FARB AND MOSHER [20], [21].

To visualise $\text{HT}(p, q)$, Figure 7 shows a compact portion of that space in the case where $p = 2$. To construct our space, we need countably many copies of each of the lines $L_k = \{z \in \mathbb{H} : \text{Im } z = q^k\}$ and strips $S_k = \{z \in \mathbb{H} : q^{k-1} \leq \text{Im } z \leq q^k\}$, where $k \in \mathbb{Z}$. These copies are pasted together in a tree-like fashion. To each vertex v of \mathbb{T} , in treebolic space there corresponds the *bifurcation line* $L_v = \{v\} \times L_k$, where $k = \mathfrak{h}(v)$. Attached below to the line L_v , there is the copy

$$S_v = \{(w, z) : w \in [v^-, v], z \in S_k, \mathfrak{h}(w) = \log_q(\text{Im } z)\}$$

of S_k . Attached above L_v , there are the strips S_u , where u ranges over the successor vertices of v (i.e., $u^- = v$).

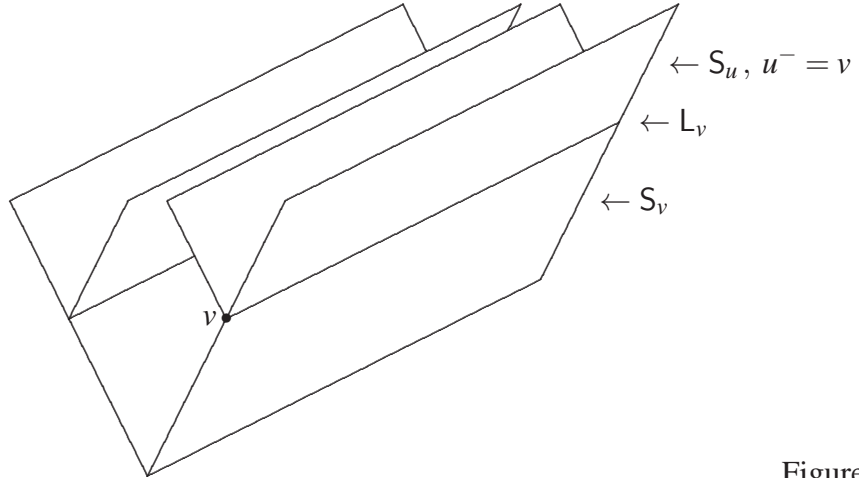


Figure 7

Thus, the sliced hyperbolic plane of Figure 6 is the front view of $\text{HT}(p, q)$, while the upside-down version of the tree of Figure 2 is the side view. In the latter picture, every bi-infinite geodesic $\pi(\mathfrak{w}, \xi)$, where $\xi \in \partial^* \mathbb{T}_p$, is the side view of one copy of \mathbb{H}_q . On each of those copies, we have the standard hyperbolic metric. It extends to $\text{HT}(p, q)$ as follows.

Let $(w_1, z_1), (w_2, z_2) \in \text{HT}$, and let $v = w_1 \wedge w_2$ (confluent with respect to \mathfrak{w} , see Figure 2). Then

$$d_{\text{HT}}((w_1, z_1), (w_2, z_2)) = \begin{cases} d_{\mathbb{H}}(z_1, z_2), & \text{if there is } \xi \in \partial^* \mathbb{T} \\ & \text{with } w_1, w_2 \in \pi(\mathfrak{w}, \xi), \\ \min\{d_{\mathbb{H}}(z_1, z) + d_{\mathbb{H}}(z, z_2) : z \in L_{\mathfrak{h}(v)}\}, & \\ \text{otherwise.} & \end{cases} \quad (2.10)$$

Indeed, in the first case, (w_1, z_1) and (w_2, z_2) belong to the common copy of \mathbb{H}_q whose side view is $\pi(\mathfrak{w}, \xi)$. In the second case, v is a vertex, and there are $\xi_1, \xi_2 \in \partial^* \mathbb{T}$ such that $\xi_1 \wedge \xi_2 = v$ and $w_i \in \pi(v, \xi_i)$, so that our points above the line L_v on two distinct hyperbolic planes that are glued together below L_v : it is necessary to pass through some point $(v, z) \in L_v$ on the way from (w_1, z_1) to (w_2, z_2) . See Figure 8.

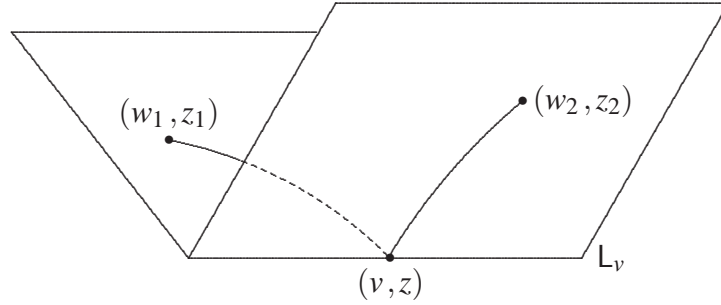


Figure 8

Using this picture, one obtains an approximate analogue of Lemma 2.8.

(2.11) Lemma. [4]. *For all $(z_1, w_1), (z_2, w_2) \in \text{HT}$, with $\delta = \log(1 + \sqrt{2})$,*

$$\begin{aligned} d_{\text{HT}}((w_1, z_1), (w_2, z_2)) &\leq d_{\mathbb{H}}(z_1, z_2) + (\log q) d_{\mathbb{T}}(w_1, w_2) - |\text{Im } z_1 - \text{Im } z_2| \\ &\leq d_{\text{HT}}((w_1, z_1), (w_2, z_2)) + 2\delta. \end{aligned}$$

Let us now describe the isometry group. We already know the group $\text{Aff}(\mathbb{T}_p)$ of all automorphisms of the tree that preserve the predecessor relation. (In terms of the action on the boundary, this is the group of all automorphisms of the tree which fix \mathfrak{w} .) On the other hand, consider the group of orientation-preserving isometries of \mathbb{H} which send the collection of all lines L_k to itself:

$$\text{Aff}(\mathbb{H}_q) = \left\{ g = \begin{pmatrix} q^n & b \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, b \in \mathbb{R} \right\} \quad \text{acting by} \quad gz = q^n z + b, \quad z \in \mathbb{H}.$$

Left Haar measure dg and the modular function $\Delta_{\text{Aff}(\mathbb{H}_q)}$ are given by

$$dg = q^{-n} dn db \quad \text{and} \quad \Delta_{\text{Aff}(\mathbb{H}_q)}(g) = q^{-n}, \quad \text{if} \quad g = \begin{pmatrix} q^n & b \\ 0 & 1 \end{pmatrix}. \quad (2.12)$$

Here, dn is counting measure on \mathbb{Z} and db is Lebesgue measure on \mathbb{R} . We can now consider the horocyclic product of $\text{Aff}(\mathbb{T}_p)$ and $\text{Aff}(\mathbb{H}_q)$. The following is not hard to prove; see [4], where the group is called $\mathcal{A}(q, p)$.

(2.13) Theorem. *The group*

$$\mathcal{B} = \mathcal{B}(p, q) = \left\{ (g_1, g_2) \in \text{Aff}(\mathbb{T}_p) \times \text{Aff}(\mathbb{H}_q) : \right. \\ \left. \log_p \Delta_{\text{Aff}(\mathbb{T}_p)}(g_1) + \log_q \Delta_{\text{Aff}(\mathbb{H}_q)}(g_2) = 0 \right\}$$

acts transitively on $\text{HT}(p, q)$ by

$$(w, z) \mapsto (g_1 w, g_2 z).$$

It is the semi-direct product

$$\mathbb{R} \rtimes \text{Aff}(\mathbb{T}_p)$$

with respect to the action

$$b \mapsto q^{\Phi(g_1)} b, \quad g_1 \in \text{Aff}(\mathbb{T}_p), \quad b \in \mathbb{R},$$

and it acts on $\text{HT}(p, q)$ with compact quotient isometric with the circle of length $\log q$. The full group of isometries of $\text{HT}(p, q)$ is generated by $\mathcal{B}(p, q)$ and the reflection

$$(w, x + iy) \mapsto (w, -x + iy).$$

As a closed subgroup of $\text{Aff}(\mathbb{T}_p) \times \text{Aff}(\mathbb{H}_q)$, the group $\mathcal{B}(p, q)$ is locally compact, compactly generated and amenable, and its modular function is given by

$$\Delta_{\mathcal{B}}(g_1, g_2) = (p/q)^{\Phi(g_1)}.$$

Again, the full isometry group is non-unimodular and cannot have a discrete, co-compact subgroup unless $q = p$.

Regarding the classification up to quasi-isometries, the following available result is not as complete as Theorem 2.6 for DL-graphs.

(2.14) Theorem. [20]. *Let $p, p' \geq 2$ be integers. Then $\text{HT}(p, p)$ is quasi-isometric with $\text{HT}(p', p')$ if and only if p and p' are powers of a common integer.*

For the general case, there is the following working hypothesis, still to be verified:¹

(2.15) Question. *Let $p, p' \geq 2$ be integers and $q, q' > 1$ real.*

Is it true that $\text{HT}(p, q)$ is quasi-isometric with $\text{HT}(p', q')$ if and only if p and p' are powers of a common integer and $\log p' / \log p = \log q' / \log q$?

¹I thank David Fisher for an exchange on this issue.

Again, there is a natural geometric compactification. Recall that the boundary of \mathbb{H} is $\mathbb{R} \cup \{\infty\}$ in the upper half plane model. The compactification $\widehat{\mathbb{H}}$ of \mathbb{H} is easier to visualise when one passes to the Poincaré disk model: it then is simply the closed unit disk. The boundary point ∞ then corresponds to the “North pole” i (the imaginary unit), while \mathbb{R} corresponds to the unit circle without i .

(2.16) Definition. *The geometric compactification $\widehat{\text{HT}}(p, q)$ is the closure of $\text{HT}(p, q)$ in $\widehat{\mathbb{T}}_p \times \widehat{\mathbb{H}}_q$, and the boundary at infinity is*

$$\partial \text{HT}(p, q) = \widehat{\text{HT}}(p, q) \setminus \text{HT}(p, q).$$

Again, we can imagine the boundary as a “filled 8” as in Figure 5, but this time the second of the two disks making up the “8” is a true unit disk: the boundary is

$$\partial \text{HT}(p, q) = \left(\widehat{\mathbb{T}}_p \times \{\infty\} \right) \cup \left(\{\varpi\} \times \widehat{\mathbb{H}}_q \right).$$

The two pieces meet in the point (ϖ, ∞) . Convergence of a sequence $\mathbf{z}_n = (w_n, z_n) \in \text{HT}(p, q)$ to the boundary is analogous to the case of DL (but recall that now the tree is a metric graph, so that convergence of a sequence to a point in \mathbb{T} does not require that the sequence stabilises at that point): When $w_n \rightarrow w \in \mathbb{T} \cup \partial^* \mathbb{T}$ then necessarily $z_n \rightarrow \infty$, whence $\mathbf{z}_n \rightarrow (w, \infty)$. In the same way, when $z_n \rightarrow z \in \mathbb{H} \cup \partial^* \mathbb{H}$, then $\mathbf{z}_n \rightarrow (\varpi, z)$. Finally, it may also occur that $w_n \rightarrow \varpi$ and $z_n \rightarrow \infty$, in which case $\mathbf{z}_n \rightarrow (\varpi, \infty)$.

C. Sol-groups, resp. manifolds

We consider again the hyperbolic upper half plane $\mathbb{H} = \{x + iw : x, w \in \mathbb{R}, w > 0\}$, but use a slightly different parametrisation and notation. The standard length element in the (x, w) -coordinates is $w^{-2}(dx^2 + dw^2)$. We pass to the *logarithmic model* by substituting $z = \log w$, and in the coordinates $(x, z) \in \mathbb{R}^2$, the length element becomes $e^{-2z}dx^2 + dz^2$. Now we also change curvature to $-p^2$ by modifying the length element into

$$ds^2 = d_p s^2 = e^{-2pz} dx^2 + dz^2.$$

We write $\mathbb{H}(p)$ for the hyperbolic plane with this parametrization and metric and $\mathbf{x} = (x, z)$ for elements of $\mathbb{H}(p)$, so that in the upper half plane model, \mathbf{x} corresponds to $x + i e^{pz}$.

The function $\mathfrak{h}(\mathbf{x}) = z$ is then (up to the scaling factor $\log p$) the Busemann function with respect to the boundary point ∞ . Thus, $(\mathbb{H}(p), \mathfrak{h})$ is a Busemann pair.

The affine group $\text{Aff}(\mathbb{H}(p)) = \{g = \begin{pmatrix} e^{pc} & a \\ 0 & 1 \end{pmatrix} : a, c \in \mathbb{R}\}$ acts on $\mathbb{H}(p)$ by $g(x, z) = (e^{pc}x + a, a + z)$ as an isometry group. Its modular function is $\Delta_p(g) = e^{-pa}$.

(2.17) Definition. For $p, q > 0$, the horocyclic product of $\mathbb{H}(p)$ and $\mathbb{H}(q)$ is the manifold

$$\text{Sol}(p, q) = \mathbb{H}(p) \times_{\mathfrak{h}} \mathbb{H}(q).$$

Topologically, it is \mathbb{R}^3 , but the length element in the 3-dimensional coordinates (x, y, z) is

$$ds^2 = d_{p,q}s^2 = e^{-2pz} dx^2 + e^{2qz} dy^2 + dz^2,$$

with the projections $(x, y, z) \mapsto (x, z) \in \mathbb{H}(p)$ and $(x, y, z) \mapsto (y, -z) \in \mathbb{H}(q)$.

It is harder to draw a reasonable picture than in the case of two trees. One should imagine to replace the two trees in Figure 4 by two hyperbolic (upper half) planes, where the second one is upside down.

Regarding the analogues of lemmas 2.8 and 2.11, so far only the following inequality has been proved, see BROFFERIO, SALVATORI AND WOESS [8].

(2.18) Lemma. [8, Proposition 2.8(iii)]. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \text{Sol}(p, q)$, then

$$\begin{aligned} d_{\text{Sol}}((x_1, y_1, z_1), (x_2, y_2, z_2)) \\ \leq d_{\mathbb{H}(p)}((x_1, z_1), (x_2, z_2)) + d_{\mathbb{H}(q)}((y_1, -z_1), (y_2, -z_2)) - |z_1 - z_2|. \end{aligned}$$

It is an open (probably not too hard) exercise to derive a matching upper bound of the form $d_{\text{Sol}}((x_1, y_1, z_1), (x_2, y_2, z_2)) + \text{const.}$

In the case of Sol , the analogue of the isometry groups $\mathcal{A}(p, q)$ for DL, resp. $\mathcal{B}(p, q)$ for HT is Sol itself. But in order to keep this analogy in mind, and also because we want to think of space and isometry group separately, we write $\mathcal{S}(p, q)$ for the corresponding Lie group.

(2.19) Facts. The Lie group

$$\mathcal{S} = \mathcal{S}(p, q) = \left\{ \mathfrak{g} = \begin{pmatrix} e^{pc} & a & 0 \\ 0 & 1 & 0 \\ 0 & b & e^{-qc} \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\}$$

can be identified with $\text{Sol}(p, q)$, such that \mathfrak{g} as above corresponds to (a, b, c) . The (isometric, fixed-point-free) action on $\text{Sol}(p, q)$ (or equivalently, the group product) is given by

$$(a, b, c) \cdot (x, y, z) = (e^{pc}x + a, e^{-qc}y + b, c + z).$$

The group is the horocyclic product of the two affine groups $\text{Aff}(\mathbb{H}(p))$ and $\text{Aff}(\mathbb{H}(q))$, consisting of all pairs (g_1, g_2) in the product of those two groups which satisfy

$$\log_p \Delta_p(g_1) + \log_q \Delta_q(g_2) = 0.$$

The modular function is

$$\Delta_{\text{Sol}(p,q)}(\mathfrak{g}) = e^{(q-p)c}, \quad \text{when} \quad \mathfrak{g} = \begin{pmatrix} e^{pc} & a & 0 \\ 0 & 1 & 0 \\ 0 & b & e^{-qc} \end{pmatrix}.$$

Again, there is no co-compact lattice in $\mathcal{S}(p, q)$ unless $p = q$. Regarding the quasi-isometry classification, we have the following analogue of Theorem 2.6.

(2.20) Theorem. [17]+[18]. $\text{Sol}(p, q)$ is quasi-isometric with $\text{Sol}(p', q')$ if and only if $p'/p = q'/q$.

Once more, there is a natural definition of the boundary of $\text{Sol}(p, q)$ when we consider our manifold as a subspace of $\mathbb{H}(p) \times \mathbb{H}(q)$. The boundary of $\mathbb{H}(p)$ is the upper boundary point ∞ together with the real line at the bottom of the upper half plane, while in the logarithmic model, the real line sits at $z = -\infty$. Anyway, it is better to think of the Poncaré disk and its compactification $\widehat{\mathbb{H}}(p)$ as a closed disk (with the proper scaling of the metric in view of the curvature parameters).

(2.21) Definition. The geometric compactification $\widehat{\text{Sol}}(p, q)$ is the closure of $\text{Sol}(p, q)$ in $\widehat{\mathbb{H}}(p) \times \widehat{\mathbb{H}}(q)$, and the boundary at infinity is

$$\partial \text{Sol}(p, q) = \widehat{\text{Sol}}(p, q) \setminus \text{Sol}(p, q).$$

The boundary looks once more like in Figure 5, but this time, both halves of the “8” are true full unit disks. This time, we omit the description of convergence to the boundary, which is completely analogous to DL and HT.

At last, we mention the work of TROYANOV [32], who has given a careful description of various features of the geometry of $\text{Sol}(1, 1)$. This includes, in particular, the *visibility boundary*. Briefly spoken, it consists of those boundary points which can be “seen” from the chosen origin (reference point) in our space as the limit of a geodesic ray that starts at the origin and converges to that boundary point. In case of $\text{Sol}(p, q)$, as a subset of the geometric boundary, the visibility boundary is the “8” without its interior points and without the point where the two circles meet. Note that this is not the same as in the visibility metric (where distance between geodesics is distance in unit sphere between their tangent vectors at 0) referred to in [32].² The visibility boundary is completely analogous for $\text{DL}(p, q)$ and $\text{HT}(p, q)$.

In this section, we have undertaken an effort to underline a variety of common

²I thank Jeremie Brieussel for an exchange on this issue.

geometric (resp. group-theoretic) features of DL, HT and Sol which become clear thanks to visualising these spaces as horocyclic products.

3 Lamplighters and other discrete subgroups

We now come to the second question of the title of this article. So far, we have seen that when $p \neq q$, then none of the groups $\mathcal{A}(p, q)$, $\mathcal{B}(p, q)$ and $\mathcal{S}(p, q)$ can contain co-compact lattices (discrete subgroups with compact quotient), and in particular, $\text{DL}(p, q)$ is far from even resembling a Cayley graph. What happens when $p = q$?

During a visit of Röggi Möller (Reykjavik) to Graz in 2000, we had discussed but not succeeded to prove that $\text{DL}(p, q)$ is not quasi-isometric with a Cayley graph. Shortly later, he sent me a letter (at that time, still on paper & by classical mail!) telling that in discussions with Peter Neumann they had realised that $\text{DL}(p, p)$ is a Cayley graph. Later we realised that this was the *lamplighter group* over \mathbb{Z} .

Let us start with an explanation in terms of graphs. Consider a finitely generated group G (resp., for a picture, one of its Cayley graphs). Imagine that at each group element (vertex) there is a lamp. Each lamp can be in p different states (off, or on in different colours or intensities) which are described by the set $\mathbb{Z}_p = \{0, \dots, p-1\}$ – the cyclic group of order p . (We might take any other finite group.) We think of G , resp. its given Cayley graph, as a street network, and imagine a lamplighter walking along. Initially, all lamps are off (state 0), and at each step the lamplighter can choose or combine the following actions: walk from a crossroad (vertex) to a neighbouring one, and/or modify the state of the lamp at the current position. After a finite number of steps, only finitely many lamps will be on. To encode this process, we have to keep track of

- the current position of the lamplighter – an element $g \in G$ (graph vertex), and
- the current configuration of lamps – a function $\eta : G \rightarrow \mathbb{Z}_p$ with finite support $\{x : \eta(x) \neq 0\}$.

Let \mathfrak{C} be the collection of all finitely supported configurations. It is a group with respect to elementwise addition mod p . We have to consider all pairs (η, g) , where $g \in G$ and $\eta \in \mathfrak{C}$. Now every $g \in G$ acts on \mathfrak{C} by $L_g \eta(x) = \eta(g^{-1}x)$. Thus, we have a semi-direct product, called the *wreath product*

$$\mathbb{Z}_p \wr G = \mathfrak{C} \rtimes G, \quad (\eta, g)(\eta', g') = (\eta + L_g \eta', gg').$$

The same works when \mathbb{Z}_p is replaced by any other group H , in which case the above addition mod p should be replaced with elementwise group operation in H . Below, we shall always have $G = \mathbb{Z}$, in which case, for $k \in \mathbb{Z}$, we have of course $L_k \eta(x) = \eta(x - k)$, and $(\eta, k)(\eta', k') = (\eta + L_k \eta', k + k')$. Wreath products are nowadays often called *lamplighter groups*, in particular when the base group is $G = \mathbb{Z}$.

The following figure illustrates an element of $\mathbb{Z}_2 \wr \mathbb{Z}$: the configuration η is $= 1$ at the \bullet s, and the lamplighter stands at the \circ .



Figure 9

We now explain the correspondence between the lamplighter group $\mathbb{Z}_p \wr \mathbb{Z}$ and the graph $DL(p, p)$. For this purpose, let us again look at Figure 2. Given any vertex of \mathbb{T}_p , we can label the edges to its successors from left to right with the digits $0, \dots, p - 1$ ($0, 1$ in Figure 2). We let Σ_p be the collection of all sequences $(\sigma(n))_{n \leq 0}$ with finite support $\{n : \sigma(n) \neq 0\}$. With a vertex x we can then associate the sequence $\sigma_x \in \Sigma_p$ of the labels on the geodesic $\pi(\bar{\omega}, x)$ coming down from $\bar{\omega}$. Given $\sigma \in \Sigma_p$ and $k \in \mathbb{Z}$, there is precisely one vertex x on the horocycle H_k such that $\sigma_x = \sigma$. In other words, we have a bijection

$$T \leftrightarrow \Sigma_p \times \mathbb{Z}, \quad \text{where} \quad x \mapsto (\sigma_x, k)$$

For example, the vertex x in Figure 2 corresponds to (σ, k) , where $k = 0$ and $\sigma = (\dots, 0, 0, 0, 1, 1)$. In the above identification, the predecessor vertex of any (σ, k) is $(\sigma', k - 1)$, where $\sigma'(n) = \sigma(n - 1)$ for all $n \leq 0$.

Now let $(\eta, k) \in \mathbb{Z}_p \wr \mathbb{Z}$. We split η at k by defining $\eta_k^- = \eta|_{(-\infty, k]}$ and $\eta_k^+ = \eta|_{[k+1, \infty)}$, both written as sequences over the non-positive integers which belong to Σ_p :

$$\eta_k^- = (\eta(k+n))_{n \leq 0} \quad \text{and} \quad \eta_k^+ = (\eta(k+1-n))_{n \leq 0}.$$

Then $x_1 = (\eta_k^-, k)$ and $x_2 = (\eta_k^+, -k)$ are vertices, one in each of the two copies of \mathbb{T}_p that make up $DL(p, p)$. This yields the correspondence between (the vertex set of) $DL(p, p)$ and the lamplighter group $\mathbb{Z}_p \wr \mathbb{Z}$. It is a rather straightforward exercise to work out that under this identification, our group acts transitively and without fixed points on $DL(p, p)$ and that the action preserves the neighbourhood relation of the graph. See [35], where this is explained in more detail.

For $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_p$, let $\delta_k^\ell \in \mathfrak{C}$ be the configuration with value ℓ at k and 0 elsewhere. Then we can subsume the preceding explanations as follows.

(3.1) Proposition. *The lamplighter group $\mathbb{Z}_p \wr \mathbb{Z}$ embeds as a discrete, co-compact subgroup into the group $\mathcal{A}(p, p)$ of Proposition 1.6. The Diestel-Leader graph $\text{DL}(p, p)$ is the Cayley graph of the lamplighter group with respect to the symmetric set of generators*

$$\{(\delta_1^\ell, 1), (\delta_0^\ell, -1) : \ell \in \mathbb{Z}_q\}.$$

This means that the actions of the lamplighter that correspond to crossing an edge in $\text{DL}(p, p)$ are: “either make first a step to the right and then switch the lamp at the arrival point to any of the possible states, or else first switch the lamp at the departure point to any of the possible states and then make a step to the left.”

Regarding treebolic space and the group $\mathcal{B}(p, p)$ of Theorem 2.13, we have the following.

(3.2) Proposition. *For integer $p \geq 2$, the Baumslag-Solitar group*

$$BS(p) = \left\{ \begin{pmatrix} p^m & k/p^l \\ 0 & 1 \end{pmatrix} : k, l, m \in \mathbb{Z} \right\} = \langle a, b \mid ab = b^p a \rangle$$

embeds as a discrete, co-compact subgroup into the group $\mathcal{B}(p, p)$.

We omit the explanation; see [20], and in more detail & closer to the spirit of the present survey, [4, §2].

Finally, we consider the Sol case and exhibit discrete, co-compact subgroups of $\mathcal{S}(p, p)$. We include an explanation because this is so obvious to the specialists that it is not too easy to find in the relevant literature.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, an integer matrix with determinant 1. We require that it has trace $a + d > 2$. Thus, it has eigenvalues $\lambda = \lambda(A) > 1$ and $1/\lambda$. Then A induces an action of \mathbb{Z} on \mathbb{Z}^2 , such that $m \in \mathbb{Z}$ acts by

$$\begin{pmatrix} k \\ l \end{pmatrix} \mapsto A^m \begin{pmatrix} k \\ l \end{pmatrix}.$$

This gives rise to the semi-direct product group

$$\mathbb{Z}^2 \rtimes_A \mathbb{Z} = \left\{ \begin{pmatrix} A^m & k \\ 0 & 0 & 1 \end{pmatrix} : k, l, m \in \mathbb{Z} \right\}. \quad (3.3)$$

We can find a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{R})$ that diagonalises A , that is,

$$A \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

If we move the lattice \mathbb{Z}^2 by that matrix, then we end up in $\mathcal{S}(p, p)$, where $p = \log \lambda$. Indeed, conjugating with the matrix

$$B = \begin{pmatrix} \alpha & 0 & \beta \\ \gamma & 0 & \delta \\ 0 & 1 & 0 \end{pmatrix},$$

we compute

$$B^{-1} \begin{pmatrix} A^m & k \\ & l \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} e^{pm} & \delta k - \beta l & 0 \\ 0 & 1 & 0 \\ 0 & -\gamma k + \alpha l & e^{-pm} \end{pmatrix}$$

Note that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and p cannot be chosen independently. Again, we subsume.

(3.4) Proposition. *For any matrix $A \in SL_2(\mathbb{Z})$ with trace > 2 , the group $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ of (3.3) embeds isomorphically into $\mathcal{S}(p, p)$ as a discrete, co-compact subgroup, where $p = \log \lambda$, and λ is the eigenvalue of A with $\lambda > 1$.*

Thus, the group acts on $\text{Sol}(p, p)$ with compact quotient.

4 Further developments

My own interest focusses on issues like random walks on graphs and groups, the associated harmonic functions and the spectral theory of the corresponding transition operators, resp. adjacency matrices. In case of non-discrete structures, it is natural to replace random walks with variants of Brownian motion. What makes me most happy is when I can use a good understanding of the geometry of the given structure to derive results in this direction.

Lamplighter groups have been of increasing interest in the context of random walks since their first appearance in this field of research in the seminal paper by KAIMANOVICH AND VERSHIK [27]. Currently, insertion of the word “lamplighter” in *MathSciNet* yields a response of 44 articles.

Realising the classical lamplighter groups $\mathbb{Z}_p \wr \mathbb{Z}$ in terms of DL graphs enhanced the interest to study random walks on $\text{DL}(p, q)$ for arbitrary integers $p, q \geq 2$. The asymptotics in space and time of random walks on $\text{DL}(p, q)$ were first studied by BERTACCHI [6].

Regarding random walk on $\mathbb{Z}_p \wr \mathbb{Z}$ – corresponding to simple random walk on $\text{DL}(p, p)$ – without using the DL description, GRIGORCHUK AND ŽUK [22] were the first to show that the spectrum is pure point, when $p = 2$, then generalised to

arbitrary p by DICKS AND SCHICK [15]. While pure point spectrum (that is, the given self-adjoint operator admits a complete orthonormal system of – typically finitely supported – eigenfunctions) is familiar in the context of fractals, this was the first example of this type regarding an infinite, finitely generated group. Using the horocyclic product structure, BARTHOLDI AND WOESS [2] provide a direct, explicit construction of the spectrum of \mathcal{A} -invariant nearest neighbour random walk on arbitrary $DL(p, q)$. It is again pure point, and it can be used to determine the exact asymptotics of return probabilities (see also REVELLE [28] for these asymptotics on the lamplighter group).

The other issue that could be treated in a rather complete way by using the horocyclic product geometry concerns positive harmonic functions and the Martin boundary, see BROFFERIO AND WOESS [35], [9], [10].

A very similar approach, though comprising several different technical details, applies to Brownian motion on the two sister structures, Sol and HT – with increasing level of difficulty. For those two, spectrum as well as Martin boundary are not yet determined rigorously, although the DL case of [9] leads to very clear ideas how the Martin compactification should look like: in the drift-free case it should be the respective geometric compactification, as described in §2, while otherwise it should be its refinement in terms of horo-levels; compare with [9]. One should also mention here the recent work on the harmonic measure of discrete time random walks on $Sol(1, 1)$ by BRIEUSSEL AND TANAKA [7].

While Sol has a smooth structure, treebolic space has singularities along all the bifurcation lines. This makes the rigorous construction of a Laplace operator (with vertical drift term) and the associated Laplace operator considerably harder, see [3]. Once this is achieved, still with some additional difficulties in view of the spatial singularities, one can proceed in a similar spirit as for random walk on DL-graphs. The results on $HT(p, q)$ concern once more rate of escape, central limit theorem, convergence to the boundary and positive harmonic functions. See [4], [5].

One common feature in all three cases is that every positive harmonic function f for the respective transition, resp. Laplace operator decomposes as

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2),$$

where x_1 and x_2 are the “coordinates” (with $\mathfrak{h}(x_1) \pm \mathfrak{h}(x_2) = 0$) in the two factors of the horocyclic product, and each f_i is a non-negative harmonic function for the projection of the respective operator on the respective factor in that product.

Of course, there are more general types of horocyclic, resp. horospherical products than the tree sister structures of §2. One is the horocyclic product of more than 2

trees,

$$\mathrm{DL}(p_1, \dots, p_d) = \{(x_1, \dots, x_d) \in \mathbb{T}_{p_1} \times \dots \times \mathbb{T}_{p_d} : \mathfrak{h}(x_1) + \dots + \mathfrak{h}(x_d) = 0\},$$

equipped with a suitable neighbourhood relation. Automorphism group, spectrum, Poisson boundary and other issues have been studied by BARTHOLDI, NEUHAUSER AND WOESS [1]. Again, the spectrum is pure point, and again, $\mathrm{DL}(p_1, \dots, p_d)$ is not a Cayley graph when the p_i do not coincide. For three trees, $\mathrm{DL}(p, p, p)$ is a Cayley graph of a finitely presented lamplighter-like group that has also been studied by CLEARY AND RILEY [12]. (The dead-end property studied in that paper and its predecessor by CLEARY AND TABACK [13] becomes immediately clear when one realises these groups in terms of DL graphs.) For $d \geq 4$ factors, in [1] a large number of cases is determined where $\mathrm{DL}(p, \dots, p)$ is a Cayley graph. The smallest case when this is not known is $\mathrm{DL}(2, 2, 2, 2)$, while $\mathrm{DL}(p, p, p, p)$ is shown to be a Cayley graph for all odd $p \geq 3$.

Given a tree with degree $p + q$, where $p, q \geq 2$, one can draw it such that every vertex has p predecessors and q successors. It also has a natural level function \mathfrak{h} , which in reality is not the Busemann function with respect to some boundary point. One can then consider the horocyclic product with “sliced” hyperbolic plane \mathbb{H}_r (where $1 < r \in \mathbb{R}$) to obtain a version of treebolic space where the strips ramify in both vertical directions. When p and q are relatively prime and r is chosen appropriately, the *non-amenable Baumslag-Solitar group* $\langle a, b \mid ab^q = b^p a \rangle$ acts on that horocyclic product as a discrete isometry group and with compact quotient. This fact is used by CUNO AND SAVA [14] in order to determine the Poisson boundary of random walks on that group.

Finally, KAIMANOVICH AND SOBIECZKY [25], [26] have constructed horocyclic products of random trees and studied random walks in the resulting random environment.

Many further interesting classes of horocyclic, resp. horospherical products are at hand and waiting for future exploration. In conclusion, let me come back to a new formulation of the question posed at the beginning, apparently still open:

Is there a (connected, locally finite, infinite) vertex-transitive graph with unimodular automorphism group that is not quasi-isometric with some Cayley graph?

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